On the investigation of summable series *

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§19 If the sum of the series, in the terms of which the undetermined quantity x is contained, was known, which will therefore be a function of x, then, whichever value is attributed to x, one will always be able to assign the sum of the series. Therefore, if instead of x one puts $x + dx$, the sum of the resulting series will equal to the sum of the first together with the differential: hence, it follows that the differential of the sum will be $=$ to the differential of the series. Because this way so the sum as the single terms will be multiplied by dx, if everywhere one divides by dx, one will have a new series, whose sum will be known. In similar manner, if this series is differentiated again and it is divided by dx everywhere, a new series will arise together with its sum and so from one summable series involving the undetermined quantity x by means of continued differentiation innumerable new equally summable series will be found.

§20 That these things are better understood, let the undetermined geometric progression be propounded, whose sum is of course known,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.}$$

If now the derivative is taken, it will be

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$$\frac{dx}{(1-x)^2} = dx + 2xdx + 3x^2dx + 4x^3dx + 5x^4dx + \text{etc.}$$

and having divided by dx one will have

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \text{etc.}$$

If one differentiates again and divides by dx, it will arise

$$\frac{2}{(1-x)^3} = 2 + 2 \cdot 3x + 3 \cdot 4x^2 + 4 \cdot 5x^3 + 5 \cdot 6x^4 + \text{etc.}$$

or

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \text{etc.}$$

where the coefficients are the triangular numbers. If one differentiates further and divides by $3dx$, one will obtain

$$\frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \text{etc.},$$

whose coefficients are the first pyramidal numbers. And, by proceeding further this way, the same series arise, which are known to arise from the expansion of the fraction $\frac{1}{(1-x)^n}$.

§21 This investigation will extend even further, if, before the differentiation is done, the series itself together with the sum is multiplied by a certain power of x or a function. So, because it is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \text{etc.},$$

multiply by x^m everywhere and it will be

$$\frac{x^m}{1-x} = x^m + x^{m+1} + x^{m+2} + x^{m+3} + x^{m+4} + \text{etc.}$$

Now differentiate this series and having divided by dx it will be

$$\frac{mx^{m-1} - (m-1)x^m}{(1-x)^2} = mx^{m-1} + (m+1)x^m + (m+2)x^{m+1} + (m+3)x^{m+2} + \text{etc.}$$

Now divide by x^{m-1}; one will have

$$\frac{m-(m-1)x}{(1-x)^2} = \frac{m}{1-x} + \frac{x}{(1-x)^2} = m + (m+1)x + (m+2)x^2 + \text{etc.}$$

Multiply this, before a new derivative is taken, by x^n that it is

$$\frac{mx^n}{1-x} + \frac{x^{n+1}}{(1-x)^2} = mx^n + (m+1)x^{n+1} + (m+2)x^{n+2} + \text{etc.}$$

Now, do the differentiation and having divided by dx it will be

$$\frac{mnx^{n-1}}{1-x} + \frac{(m+n+1)x^n}{(1-x)^2} + \frac{2x^{n+1}}{(1-x)^3}$$
$$= mnx^{n-1} + (m+1)(n+1)x^n + (m+2)(n+2)x^{n+1} + \text{etc.}$$

But having divided by x^{n-1} it will be

$$\frac{mn}{1-x} + \frac{(m+n+1)x}{(1-x)^2} + \frac{2xx}{(1-x)^3}$$
$$mn + (m+1)(n+1)x + (m+2)(n+2)x^2 + \text{etc.}$$

and it will be possible to proceed further this way; but one will always find the same series which arise from the expansions of the fractions constituting the sum.

§22 Since the sum of the geometric progression assumed at first can be assigned up to any certain term, this way also series consisting of finite a number of terms will be summed. Because it is

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n,$$

it will be having done the differentiation and having divided by dx

$$\frac{1}{(1-x)^2} - \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}.$$

Hence, the sum of the powers of natural numbers up to a certain term can be found. For, multiply this series by x that it is

$$\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}=x+2x^2+3x^3+\cdots+nx^n,$$

which differentiated again and divided by dx will give

$$\frac{1+x-(n+1)x^n+(2nn+2n-1)x^{n+1}-nnx^{n+2}}{(1-x)^3}=1+4x+9x^2+\cdots+n^2x^{n-1};$$

this multiplied by x will give

$$\frac{x+x^2-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}=x+4x^2+9x^3+\cdots+n^2x^n,$$

which differentiated, divided by dx and multiplied by x will produce this series

$$x+8x^2+27x^3+\cdots+n^2x^n,$$

whose sum therefore will be found. And from this in similar manner the indefinite sum of the bisquares and higher powers will be found.

§23 Therefore, this method can be applied to all series containing an undetermined quantity whose sum is known, of course. Because except geometric series all recurring series enjoy the same prerogatives that they can be summed not only up to infinity but also to any given term, one will be able to also find innumerable other summable series from these by the same method. Because most extensive work would be necessary, if we wanted to follow up on this, let us consider only one single case.

Let this series be propounded

$$\frac{x}{1-x-xx}=x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+\text{etc.},$$

which differentiated and divided by dx will give

$$\frac{1+xx}{(1-x-xx)^2}=1+2x+6x^2+12x^3+25x^4+48x^5+91x^6+\text{etc.}$$

But it easily becomes clear that all series resulting this way will also be recurring whose sums can even be found from their nature itself.

§24 Therefore, in general, if the sum of a certain series contained in this form

$$ax + bx^2 + cx^3 + dx^4 + \text{etc.}$$

was known which sum we want to put $= S$, one will be able to find the sum of the same series, if the single terms are each multiplied by terms of an arithmetic progression. For, let

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.};$$

multiply by x^m; it will be

$$Sx^m = ax^{m+1} + bx^{m+2} + cx^{m+3} + dx^{m+4} + \text{etc.};$$

differentiate this equation and divide by dx

$$mSx^{m-1} + x^m \frac{dS}{dx} = (m+1)ax^m + (m+2)bx^{m+1} + (m+3)cx^{m+2} + \text{etc.};$$

divide by x^{m-1} and it will be

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \text{etc.}$$

Therefore, if one desires the sum of this following series

$$\alpha ax + (\alpha + \beta)bx^2 + (\alpha + 2\beta)cx^3 + (\alpha + 3\beta)dx^4 + \text{etc.},$$

multiply the superior by β and put $m\beta + \beta = \alpha$ that it is $M = \frac{\alpha - \beta}{\beta}$ and the sum of this series will be

$$= (\alpha - \beta)S + \frac{\beta xdS}{dx}.$$

§25 One will also be able to find the sum of this propounded series, if its single terms are each multiplied by terms of series of second order, whose second differences are just constant, of course. For, because we already found

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \text{etc.},$$

multiply by x^n that it is

$$mSx^n + \frac{x^{n+1}dS}{dx} = (m+1)ax^{n+1} + (m+2)bx^{n+2} + \text{etc.};$$

differentiate having put dx constant and divide by dx

$$mnSx^{n-1} + \frac{(m+n+1)x^nS}{dx} + \frac{x^{n+1}ddS}{dx^2}$$
$$= (m+1)(n+1)ax^n + (m+2)(n+2)bx^{n+1} + \text{etc.}$$

Divide by x^{n-1} and multiply by k that it is

$$mnkS + \frac{(m+n+1)kxdS}{dx} + \frac{kx^2ddD}{dx^2}$$
$$= (m+1)(n+1)kax + (m+2)(n+2)kbx^2 + (m+3)(n+3)kcx^3 + \text{etc.}$$

Now, compare this series to that one; it will be

								Diff. I							Diff. II	
kmn	$+1km$	$+1kn$	$+1k$	$=\alpha$												
								km	$+kn$	$+3k$	$=\beta$					
knm	$+2km$	$+2kn$	$+4k$	$=\alpha$	$+1\beta$										$2k$	$=\gamma$
								km	$+kn$	$+5k$	$=\beta$	$+\gamma$				
lnm	$+3km$	$+3kn$	$+9k$	$=\alpha$	$+2\beta$	$+\gamma$										

Therefore, $k = \frac{1}{2}\gamma$ and $m + n = \frac{2\beta}{\gamma} - 3$ and

$$mn = \frac{\alpha}{k} - m - n - 1 = \frac{2\alpha}{\gamma} - \frac{2\beta}{\gamma} + 2 = \frac{2(\alpha - \beta + \gamma)}{\gamma}.$$

Hence, the sum of the series in question will be

$$(\alpha - \beta + \gamma)S + \frac{(\beta - \gamma)xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2}.$$

§26 In similar manner, one will be able to find the sum of this series

$$Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.},$$

if the sum S of this series was known of course

$$S = a + bx + cx^2 + dx^3 + dx^3 + ex^4 + fx^5 + \text{etc.}$$

and A, B, C, D etc. constitute a series which is led to constant differences. For, since it its form is concluded from the preceding, assume this sum

$$\alpha S + \frac{\beta x dS}{dx} + \frac{\gamma x^2 ddS}{2dx^2} + \frac{\Delta x^3 d^3 S}{6dx^3} + \frac{\varepsilon x^4 d^4 S}{24dx^4} + \text{etc.}$$

Now, to find the letters α, β, γ, δ etc., expand the single series and it will be

$$\begin{aligned}
\alpha S &= 1\alpha a + 1\alpha bx + 1\alpha cx^2 + 1\alpha dx^3 + 1\alpha ex^4 + \text{etc.} \\
\frac{\beta x dS}{dx} &= \quad + 1\beta bx + 2\beta cx^2 + 3\beta dx^3 + 4\beta ex^4 + \text{etc.} \\
\frac{\gamma x^2 ddS}{2dx^2} &= \quad + 1\gamma cx^2 + 3\gamma dx^3 + 6\gamma ex^4 + \text{etc.} \\
\frac{\delta x^3 d^3 S}{6dx^3} &= \quad + 1\delta dx^3 + 4\delta ex^4 + \text{etc.} \\
\frac{\varepsilon x^4 d^4 S}{24dx^4} &= \quad + 1\varepsilon ex^4 + \text{etc.} \\
&\qquad \text{etc.;}
\end{aligned}$$

compare this collected together to the propounded one

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.}$$

and having done the comparison of the single terms

$$\begin{aligned}
\alpha &= A \\
\beta &= B - \alpha = B - A \\
\gamma &= C - 2\beta - \alpha = C - 2B + A \\
\Delta &= D - 3\gamma - 3\beta - \alpha = D - 3C + 3B - A \\
&\text{etc.}
\end{aligned}$$

Having found these values the sought after sum will therefore be

$$Z = AS + \frac{(B-A)xdS}{1dx} + \frac{(C-2B+A)x^2 ddS}{1 \cdot 2dx^2} + \frac{(D-3C+3B-A)x^3 d^3 S}{1 \cdot 2 \cdot 3dx^3} + \text{etc.,}$$

or if the differences of the series A, B, C, D, E etc. are indicated in the usual manner, it will be

$$Z = AS + \frac{\Delta A \cdot xdS}{1dx} + \frac{\Delta^2 A \cdot x^2 d^2 S}{1 \cdot 2dx^2} + \frac{\Delta^3 A \cdot x^3 d^3 S}{1 \cdot 2 \cdot 3dx^3} + \text{etc.}$$

if it was, as we assumed,

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \text{etc.}$$

If therefore the series A, B, C, D etc. finally has constant differences, one will be able to express the sum of the series Z in finite terms.

§27 Since having taken e for the number whose hyperbolic logarithm is $= 1$ it is

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{etc.};$$

assume this series for the first, and because it is $S = e^x$, it will be $\frac{dS}{dx} = e^x$, $\frac{ddS}{dx^2} = e^x$ etc. Therefore, the sum of this series which is composed from that one and this one A, B, C, D etc.

$$A + \frac{Bx}{1} + \frac{Cx^2}{1 \cdot 2} + \frac{Dx^3}{1 \cdot 2 \cdot 3} + \frac{Ex^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

will be expressed this way

$$e^x \left(A + \frac{x\Delta A}{1} + \frac{xx\Delta^2 A}{1 \cdot 2} + \frac{x^3 \Delta^3 A}{1 \cdot 2 \cdot 3} + \frac{x^4 \Delta^4 A}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.} \right).$$

So, if this series is propounded

$$2 + \frac{5x}{1} + \frac{10x^2}{1 \cdot 2} + \frac{17x^2}{1 \cdot 2 \cdot 3} + \frac{26x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{37x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{etc.},$$

because of the series

$$A, \quad B, \quad C, \quad D, \quad E \quad \text{etc.}$$

$$\begin{array}{lllllllllll} A & = 2, & & 5, & & 10, & & 17, & & 26 & \text{etc.} \\ \Delta A & = & 3, & & 5, & & 7, & & 9 & \text{etc.} \\ \Delta\Delta A & = & & 2, & & 2, & & 2 & \text{etc.} \end{array}$$

the sum of this series

$$2+5x+\frac{10x^2}{2}+\frac{17x^3}{6}+\frac{26x^4}{24}+\text{etc.}$$

will be

$$=e^x(2+3x+xx)=e^x(1+x)(2+x),$$

which is immediately clear. For, it is

$$\begin{array}{llllllll} 2e^x & =2+\frac{2x}{1} & +\frac{2x^2}{2} & +\frac{2x^3}{6} & +\frac{2x^4}{24} & +\text{etc.} \\ 3xe^x & = \quad +3x & +\frac{3x^2}{1} & +\frac{3x^2}{2} & +\frac{3x^4}{6} & +\text{etc.} \\ xxe^x & = & +xx & +\frac{x^3}{1} & +\frac{x^4}{2} & +\text{etc.} \end{array}$$

and in total

$$e^x(1+3x+xx)=2+5x+\frac{10xx}{2}+\frac{17x^3}{6}+\frac{24x^4}{24}+\text{etc.}$$

§28 The things treated up to now not only concern series running to an infinite number of terms, but also sums of a finite number of terms; for, the coefficients a, b, c, d etc. can either proceed to infinity or can be truncated whereever one desires. But because this does not demand any further explanation, let us consider in more detail what follows from the things mentioned up to now. Therefore, having propounded any arbitrary series, whose single terms shall consist of two factors, the one of which shall constitute a series leading to constant differences, one will be able to assign the sum of this series, as long as having omitted these factors the sum was summable. Of course, if this series is propounded

$$Z=Aa+Bbx+Ccx^2+Ddx^3+Eex^4+\text{etc.}$$

in which the quantities A, B, C, D, E etc. constitute a series of such a kind which is finally led to constant differences, then one will be able to exhibit the sum of this series, as long as one has the sum S of this series

$$S = a + bx + cx^2 + dx^3 + ex^4 + \text{etc.}$$

For, having taken the continued differences from the progression A, B, C, D, E etc., as we showed at the beginning of this book,

$$\begin{array}{ccccccccccccc}
A, & & B, & & C, & & D, & & E, & & F, & & \text{etc.} \\
& \Delta A & & \Delta B, & & \Delta C, & & \Delta D, & & \Delta E & & \text{etc.} & \\
& & \Delta^2 A & & \Delta^2 B, & & \Delta^2 C, & & \Delta^2 D & & \text{etc.} & & \\
& & & \Delta^3 A & & \Delta^3 B, & & \Delta^3 C, & & \text{etc.} & & & \\
& & & & \Delta^4 A & & \Delta^4 B, & & \text{etc.} & & & & \\
& & & & & \Delta^5 A & & \text{etc.} & & & & & \\
& & & & & & \text{etc.} & & & & & &
\end{array}$$

the sum of the propounded series will be

$$Z = SA + \frac{xdS}{1dx}\Delta A + \frac{x^2 ddS}{1 \cdot 2dx^2}\Delta^2 A + \frac{x^3 d^3 S}{1 \cdot 2 \cdot 3dx^3}\Delta^3 A + \text{etc.}$$

after having put dx constant in the higher powers of S.

§29 If therefore the series A, B, C, D etc. never leads to constant differences, the sum of the series Z will be expressed by means of a new infinite series which will converge more than the propounded one, and so this series will be transformed into another equal one. To illustrate this let this series be propounded

$$Y = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \frac{y^5}{5} + \frac{y^6}{6} + \text{etc.},$$

which is known to express $\ln \frac{1}{1-y}$ such that it is $Y = -\ln(1-y)$. Divide the series by y and put $y = x$ and $Y = yZ$ that it is

$$Z = -\frac{1}{y}\ln(1-y) = -\frac{1}{x}\ln(1-x);$$

it will be

$$Z = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \text{etc.},$$

which compared to this one

$$S = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.} = \frac{1}{1-x}$$

will give these values for the series A, B, C, D, E etc.

$$\begin{array}{ccccccccccc}
1, & & \frac{1}{2}, & & \frac{1}{3}, & & \frac{1}{4}, & & \frac{1}{5} & & \text{etc.} \\
& -\frac{1}{1\cdot 2}, & & -\frac{1}{2\cdot 3}, & & -\frac{1}{3\cdot 4}, & & -\frac{1}{4\cdot 5} & & \text{etc.} & \\
& & \frac{1\cdot 2}{1\cdot 2\cdot 3}, & & \frac{1\cdot 2}{2\cdot 3\cdot 4}, & & \frac{1\cdot 2}{3\cdot 4\cdot 5} & & \text{etc.} & & \\
& & & -\frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4}, & & -\frac{1\cdot 2\cdot 3}{2\cdot 3\cdot 4\cdot 5} & & \text{etc.} & & & \\
& & & & \text{etc.} & & & & & &
\end{array}$$

Therefore, it will be

$$A = 1, \quad \Delta A = -\frac{1}{2}, \quad \Delta^2 A = \frac{1}{3}, \quad \Delta^3 A = -\frac{1}{4} \quad \text{etc.}$$

Further, because it is $S = \frac{1}{1-x}$, it will be

$$\frac{dS}{dx} = \frac{1}{(1-x)^2}, \quad \frac{ddS}{1\cdot 2dx^2} = \frac{1}{(1-x)^3}, \quad \frac{d^3S}{1\cdot 2\cdot 3dx^3} = \frac{1}{(1-x)^4} \quad \text{etc.}$$

Having substituted these values this sum will arise

$$Z = \frac{1}{1-x} - \frac{x}{2(1-x)^2} + \frac{x^2}{3(1-x)^3} - \frac{x^3}{4(1-x)^4} + \frac{x^4}{5(1-x)^5} - \text{etc.}$$

Therefore, because it is $x = y$ and $Y = -\ln(1-y) = yZ$, it will be

$$-\ln(1-y) = \frac{y}{1-y} - \frac{y^2}{2(1-y)^2} + \frac{y^3}{3(1-y)^3} - \frac{y^4}{4(1-y)^4} + \text{etc.},$$

which series manifestly expresses $\ln\left(1 + \frac{y}{1-y}\right) = \ln\frac{1}{1-y} = -\ln(1-y)$, the validity of which is even clear by means of the things demonstrated before.

§30 Now let this series be propounded that the use becomes also clear, if only odd powers occur and the signs alternate,

$$Y = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \frac{y^9}{9} - \frac{y^{11}}{11} + \text{etc.},$$

from which it is known that it is $Y = \arctan y$.

Divide this series by y and put $\frac{Y}{y} = Z$ and $yy = x$; it will be

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \text{etc.}$$

If it is compared to this one

$$S = 1 - x + xx - x^3 + x^4 - x^5 + \text{etc.},$$

it will be $S = \frac{1}{1+x}$ and the series of coefficients A, B, C, D etc. will become

$$\begin{array}{lllllllll}
A = 1, & & \frac{1}{3}, & & \frac{1}{5}, & & \frac{1}{7}, & & \frac{1}{9}, \quad \text{etc.} \\
\Delta A = & -\frac{2}{3}, & & -\frac{2}{3\cdot 5}, & & -\frac{2}{5\cdot 7}, & & -\frac{2}{7\cdot 9} & \text{etc.} \\
\Delta^2 A = & & \frac{2\cdot 4}{3\cdot 5}, & & \frac{2\cdot 4}{3\cdot 5\cdot 7}, & & \frac{2\cdot 4}{5\cdot 7\cdot 9} & & \text{etc.} \\
\Delta^3 A = & & & -\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}, & & -\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 9} & & \text{etc.} & \\
\Delta^4 A = & & & & \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9} & \text{etc.} & & & \\
 & & & & \text{etc.} & & & &
\end{array}$$

But because it is $S = \frac{1}{1+x}$, it will be

$$\frac{dS}{1dx} = -\frac{1}{(1+x)^2}, \quad \frac{ddS}{1\cdot 2dx^2} = \frac{1}{(1+x)^3}, \quad \frac{d^3S}{1\cdot 2\cdot 3dx^3} = -\frac{1}{(1+x)^4} \quad \text{etc.}$$

Hence, having substituted these values, the form will become

$$Z = \frac{1}{1+x} + \frac{2x}{3(1+x)^2} + \frac{2\cdot 4x^2}{3\cdot 5(1+x)^3} + \frac{2\cdot 4\cdot 6x^3}{3\cdot 5\cdot 7(1+x)^4} + \text{etc.}$$

having resubstituted $x = yy$ and multiplied by y it will become

$$Y = \arctan y = \frac{y}{1+yy} + \frac{2y^3}{3(1+yy)^2} + \frac{2\cdot 4y^5}{3\cdot 5(1+yy)^3} + \frac{2\cdot 4\cdot 6y^7}{3\cdot 5\cdot 7(1+yy)^4} + \text{etc.}$$

§31 One can also transform the superior series by means of which the arc of a circle is expressed another way by comparing it to the logarithmic series.

For, let us consider the series

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \text{etc.},$$

which we want to compare to this one

$$S = \frac{1}{0} - \frac{x}{2} + \frac{xx}{4} - \frac{x^3}{6} + \frac{x^4}{8} - \text{etc.} = \frac{1}{0} - \frac{1}{2}\ln(1+x),$$

and the values of the letters A, B, C, D etc. will be

$$\begin{array}{lccccccccccc}
A & = & \frac{0}{1}, & & \frac{2}{3}, & & \frac{4}{5}, & & \frac{6}{7}, & & \frac{8}{9} & \text{etc.} \\
\Delta A & = & & \frac{2}{3}, & & \frac{+2}{3\cdot 5}, & & \frac{+2}{5\cdot 7}, & & \frac{+2}{7\cdot 9} & & \text{etc.} \\
\Delta^2 A & = & & & \frac{-2\cdot 4}{3\cdot 5}, & & \frac{-2\cdot 4}{3\cdot 5\cdot 7}, & & \frac{-2\cdot 4}{5\cdot 7\cdot 9} & & \text{etc.} & \\
\Delta^3 A & = & & & & \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} & \text{etc.} & & & & & \\
 & & & & & \text{etc.} & & & & & &
\end{array}$$

Further, because it is $S = \frac{1}{0} - \frac{1}{2}\ln(1+x)$, it will be

$$\frac{dS}{1dx} = -\frac{1}{2(1+x)}, \quad \frac{ddS}{1\cdot 2dx^2} = \frac{1}{4(1+x)^2},$$

$$\frac{d^3S}{1\cdot 2\cdot 3dx^3} = -\frac{1}{6(1+x)^3}, \quad \frac{d^4S}{1\cdot 2\cdot 3\cdot 4dx^4} = \frac{1}{8(1+x)^4} \quad \text{etc.}$$

Therefore, it will be $SA = D\frac{0}{1} = 1$ and from the remaining it will be

$$Z = 1 - \frac{x}{3(1+x)} - \frac{2xx}{3\cdot 5(1+x)^2} - \frac{2\cdot 4x^3}{3\cdot 5\cdot 7(1+x)^3} - \text{etc.}$$

Now, let us put $x = yy$ and multiply by y; it will be

$$Y = \arctan y = y - \frac{y^3}{3(1+yy)} - \frac{2y^2}{3 \cdot 5(1+yy)^2} - \frac{2 \cdot 4y^7}{3 \cdot 5 \cdot 7(1+yy)^3} - \text{etc.}$$

This transformation will therefore not be impeded by the infinite term $\frac{1}{0}$ which went into the series S. But if there remains any doubt, just expand the single terms except the first into power series in y and one will discover the indeed the series first propounded arises.

§32 Up to now we considered only series of such a kind in which all powers of the variable occured. Now, we want therefore proceed to other series which in the single terms contain the same power of the variable of which kind this series is

$$S = \frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} + \text{etc.}$$

For, if the sum S of this series was known and is expressed by a certain function of x, by differentiating and by dividing by $-dx$ it will be

$$\frac{-dS}{dx} = \frac{1}{(a+x)^2} + \frac{1}{(b+x)^2} + \frac{1}{(c+x)^2} + \frac{1}{(d+x)^2} + \text{etc.}$$

If this series is further differentiated and divided by $-2dx$, one will recognize the series of the cubes

$$\frac{ddS}{2dx^2} = \frac{1}{(a+x)^3} + \frac{1}{(b+x)^3} + \frac{1}{(c+x)^3} + \frac{1}{(d+x)^3} + \text{etc.}$$

and this differentiated again and divided by $-3dx$ will give

$$\frac{-d^3S}{dx^3} = \frac{1}{(a+x)^4} + \frac{1}{(b+x)^4} + \frac{1}{(c+x)^4} + \frac{1}{(d+x)^4} + \text{etc.}$$

And in similar way, the sum of all following powers will be found, as long as the sum of the first series was known.

§33 But we found series of fractions of this kind involving an undetermined quantity above in the *Introductio*, where we showed, if the half-circumference circle, whose radius is $= 1$, is set $= \pi$, that it will be

$$\frac{\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.}$$
$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.}$$

Because it is therefore possible to assume any arbitrary numbers for m and n, let us set $n = 1$ and $m = x$ that we obtain a series similar to that one we had propounded in the preceding paragraph; having done this it will be

$$\frac{\pi}{\sin\pi x} = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \text{etc.}$$
$$\frac{\pi\cos\pi}{\sin\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

Therefore, one will be able to exhibit the sums of any powers of fractions arising from these fractions by means of differentiations.

§34 Let us consider the first series and for the sake of brevity put $\frac{\pi}{\sin\pi x} = S$, whose higher differentials shall be taken having put dx constant, and it will be

$$\begin{aligned}
S &= \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \text{etc.}\\
\frac{-dS}{dx} &= \frac{1}{xx} - \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(3+x)^2} - \frac{1}{(3-x)^2} - \text{etc.}\\
\frac{ddS}{2d^2x} &= \frac{1}{x^3} + \frac{1}{(1-x)^3} - \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(3+x)^3} + \frac{1}{(3-x)^3} - \text{etc.}\\
\frac{-d^3S}{6d^3x} &= \frac{1}{x^4} - \frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(3+x)^4} - \frac{1}{(3-x)^4} - \text{etc.}\\
\frac{d^4S}{24d^4x} &= \frac{1}{x^5} + \frac{1}{(1-x)^5} - \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(3+x)^5} + \frac{1}{(3-x)^5} - \text{etc.}\\
\frac{-d^5S}{120d^3x} &= \frac{1}{x^6} - \frac{1}{(1-x)^6} - \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(3+x)^6} - \frac{1}{(3-x)^6} - \text{etc.}
\end{aligned}$$

etc.

where it is to be noted that in the even powers the signs follow the same law and in similar way in the odd the same law of the signs is observed. Therefore, the sums of all these series are found from the differentials of the expression $S = \frac{\pi}{\sin \pi x}$.

§35 To express this differentials in a simpler way let us put

$$\sin \pi = p \quad \text{and} \quad \cos \pi = q;$$

it will be

$$dp = \pi dx \cos \pi x = \pi q dx \quad \text{and} \quad dq = -\pi p dx.$$

Because therefore it is $S = \frac{\pi}{p}$, it will be

$$\begin{aligned}
\frac{-dS}{dx} &= \frac{\pi^2 q}{pp} \\
\frac{ddS}{dx^2} &= \frac{\pi^3(pp+2qq)}{p^3} = \frac{\pi^3(qq+1)}{p^3} \quad \text{because it is} \quad pp+qq=1 \\
\frac{-d^3S}{dx^3} &= \pi^4\left(\frac{5q}{pp}+\frac{6q^3}{p^4}\right) = \frac{\pi^4(q^3+5q)}{p^4} \\
\frac{d^4S}{dx^4} &= \pi^5\left(\frac{24q^4}{p^5}+\frac{28q^2}{p^3}+\frac{5}{p}\right) = \frac{\pi^5(q^4+18q^2+5)}{p^5} \\
\frac{-d^5S}{dx^5} &= \pi^6\left(\frac{120q^5}{p^6}+\frac{180q^3}{p^4}+\frac{61q}{pp}\right) = \frac{\pi^6(q^5+58q^3+61q)}{p^6} \\
\frac{d^6S}{dx^6} &= \pi^7\left(\frac{720q^6}{p^7}+\frac{1320q^4}{p^5}+\frac{662q^2}{p^3}+\frac{61}{p}\right) = \frac{\pi^7(q^6+179q^4+479q^2+61)}{p^7} \\
\frac{-d^7S}{dx^7} &= \pi^8\left(\frac{5040q^7}{p^8}+\frac{10920q^5}{p^6}+\frac{7266q^3}{p^4}+\frac{1385q}{p^2}\right)
\end{aligned}$$

or

$$\begin{aligned}
&= \frac{\pi^8}{p^8}(q^7+543q^5+3111q^3+1385q) \\
\frac{d^8S}{dx^8} &= \pi^9\left(\frac{40320q^8}{p^9}+\frac{100800q^6}{p^7}+\frac{83664q^4}{p^5}+\frac{24568q^2}{p^3}+\frac{1385}{p}\right)
\end{aligned}$$

or

$$= \frac{\pi^9}{p^9}(q^8 + 1636q^6 + 18270q^4 + 19028q^2 + 1385)$$

etc.

These expressions are easily continued as far as one desires; for, if it was

$$\pm\frac{d^n S}{dx^n} = \pi^{n+1}\left(\frac{\alpha q^n}{p^{n+1}} + \frac{\beta q^{n-2}}{p^{n-1}} + \frac{\gamma q^{n-4}}{p^{n-3}} + \frac{\delta q^{n-6}}{p^{n-5}} + \text{etc.}\right),$$

then its differential having changed the signs will be

$$\mp\frac{d^{n+1}S}{dx^{n+1}}\left\{\begin{array}{l}(n+1)\alpha\dfrac{q^{n+1}}{p^{n+2}} + (n\alpha + (n-1)\beta)\dfrac{q^{n-1}}{p^n} + ((n-2)\beta + (n-3)\gamma)\dfrac{q^{n-3}}{p^{n-2}}\\ + ((n-4)\gamma + (n-5)\delta)\dfrac{q^{n-5}}{p^{n-4}} + \text{etc.}\end{array}\right\}$$

§36 Therefore, from these ones will obtain the following sums of the superior series exhibited in § 34

$$\begin{aligned}
S &= \pi\cdot\frac{1}{p}\\
\frac{-dS}{dx} &= \frac{\pi^2}{1}\cdot\frac{q}{p^2}\\
\frac{ddS}{24dx^2} &= \frac{\pi^3}{2}\left(\frac{2q^2}{p^3} + \frac{1}{p}\right)\\
\frac{-d^3S}{6dx^3} &= \frac{\pi^4}{6}\left(\frac{6q^3}{p^4} + \frac{5q}{p^2}\right)\\
\frac{d^4S}{24dx^4} &= \frac{\pi^5}{24}\left(\frac{24q^4}{p^5} + \frac{28q^2}{p^3} + \frac{5}{p}\right)\\
\frac{-d^5S}{120dx^5} &= \frac{\pi^6}{120}\left(\frac{120q^5}{p^6} + \frac{180q^3}{p^4} + \frac{61q}{p^2}\right)\\
\frac{d^6S}{720dx^6} &= \frac{\pi^7}{720}\left(\frac{720q^6}{p^7} + \frac{1320q^4}{p^5} + \frac{662q^2}{p^3} + \frac{61}{p}\right)
\end{aligned}$$

$$\frac{-d^7S}{720dx^6} = \frac{\pi^8}{5040}\left(\frac{5040q^7}{p^8} + \frac{10920q^5}{p^6} + \frac{7266q^3}{p^4} + \frac{1385q}{p^2}\right)$$
$$\frac{d^8S}{40320dx^8} = \frac{\pi^9}{40320}\left(\frac{40320q^8}{p^9} + \frac{100800q^6}{p^7} + \frac{83664q^4}{p^5} + \frac{24568q^2}{p^3} + \frac{1385}{p}\right)$$
etc.

§37 Let us treat the other series found above [§ 33] in similar way

$$\frac{\pi\cos\pi x}{\sin\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

and for the sake of brevity having put $\frac{\pi\cos\pi x}{\sin\pi x} = T$ the following summations will arise

$$T = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \text{etc.}$$
$$\frac{-dT}{dx} = \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} + \text{etc.}$$
$$\frac{ddT}{2dx^2} = \frac{1}{x^3} - \frac{1}{(1-x)^3} + \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^2} - \text{etc.}$$
$$\frac{-d^3T}{6d^3x} = \frac{1}{x^4} + \frac{1}{(1-x)^4} + \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} + \text{etc.}$$
$$\frac{d^4T}{24dx^4} = \frac{1}{x^5} - \frac{1}{(1-x)^5} + \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} - \text{etc.}$$
$$\frac{-d^5T}{120d^5x} = \frac{1}{x^6} + \frac{1}{(1-x)^6} + \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} + \text{etc.}$$
etc.,

where in the even powers all terms are positive, but in the odd the signs $+$ and $-$ alternate.

§38 To find the values of these differentials let us as before put

$$\sin\pi x = p \quad \text{and} \quad dq = -\pi p dx$$

that it is $pp + qq = 1$; it will be

$$dp = \pi q dx \quad \text{and} \quad dq = -\pi p dx.$$

Having added these values it will be

$$\begin{aligned}
T &= \pi \cdot \frac{q}{p} \\
\frac{-dT}{dx} &= \pi^2 \left(\frac{qq}{pp} + 1 \right) = \frac{\pi^2}{pp} \\
\frac{ddT}{dx^2} &= \pi^3 \left(\frac{2q^3}{p^3} + \frac{2q}{p} \right) = \frac{2\pi^3 q}{p^3} \\
\frac{-d^3T}{dx^3} &= \pi^4 \left(\frac{6q^4}{p^4} + \frac{8qq}{pp} + 2 \right) = \pi^4 \left(\frac{6qq}{p^4} + \frac{2}{pp} \right) \\
\frac{d^4T}{dx^4} &= \pi^5 \left(\frac{24q^3}{p^5} + \frac{16q}{p^3} \right) \\
\frac{-d^5T}{dx^5} &= \pi^6 \left(\frac{120q^4}{p^6} + \frac{120qq}{p^4} + \frac{16}{pp} \right) \\
\frac{d^6T}{dx^6} &= \pi^7 \left(\frac{720q^5}{p^7} + \frac{960q^3}{p^5} + \frac{272q}{p^3} \right) \\
\frac{-d^7T}{dx^7} &= \pi^8 \left(\frac{5040q^6}{p^8} + \frac{8400q^4}{p^6} + \frac{3696q^2}{q^4} + \frac{272}{p^2} \right) \\
\frac{d^8T}{dx^8} &= \pi^9 \left(\frac{40320q^7}{p^9} + \frac{80640q^5}{p^7} + \frac{48384q^3}{p^5} + \frac{7936q}{p^3} \right) \\
&\text{etc.}
\end{aligned}$$

These formulas can easily be continued as far as one desires. For, if it is

$$\pm \frac{d^nT}{dx^n} = \pi^{n+1} \left(\frac{\alpha q^{n-1}}{p^{n+1}} + \frac{\beta q^{n-3}}{p^{n-1}} + \frac{\gamma q^{n-5}}{p^{n-3}} + \frac{\delta q^{n-7}}{p^{n-5}} + \text{etc.} \right),$$

the expression will be the following

$$\mp \frac{d^{n+1}T}{dx^{n+1}} = \pi^{n+2} \left(\frac{(n+1)\alpha q^n}{p^{n+2}} + \frac{(n-1)(\alpha+\beta)q^{n-2}}{p^n} + \frac{(n-3)(\beta+\gamma)q^{n-4}}{p^{n-2}} + \text{etc.} \right)$$

For, if it is

$$\pm \frac{d^nT}{dx^n} = \pi^{n+1} \left(\frac{\alpha q^{n-1}}{p^{n+1}} + \frac{\beta q^{n-3}}{p^{n-1}} + \frac{\gamma q^{n-5}}{p^{n-3}} + \frac{\delta q^{n-7}}{p^{n-5}} + \text{etc.} \right),$$

the following expression will be

$$\mp\frac{d^{n+1}T}{dx^{n+1}} = \pi^{n+2}\left(\frac{(n+1)\alpha q^n}{p^{n+2}} + \frac{(n-1)(\alpha+\beta)q^{n-2}}{p^n} + \frac{(n-3)(\beta+\gamma)q^{n-4}}{p^{n-2}} + \text{etc.}\right).$$

§39 Therefore, the series of powers given in § 37 will have the following sums having put $\sin\pi x = p$ and $\cos\pi x = q$

$$\begin{aligned}
T &= \pi\cdot\frac{q}{p}\\
\frac{-dT}{dx} &= \pi^2\frac{1}{pp}\\
\frac{ddT}{2dx^2} &= \pi^3\frac{q}{p^3}\\
\frac{-d^3T}{6dx^3} &= \pi^4\left(\frac{qq}{\pi^4} + \frac{1}{3pp}\right)\\
\frac{-d^4T}{24dx^4} &= \pi^5\left(\frac{q^3}{p^5} + \frac{2q}{3p^3}\right)\\
\frac{-d^5T}{120dx^5} &= \pi^6\left(\frac{q^4}{p^6} + \frac{3qq}{p^4} + \frac{2}{15pp}\right)\\
\frac{d^6T}{720dx^6} &= \pi^7\left(\frac{q^5}{p^7} + \frac{4q^3}{3p^5} + \frac{17q}{45p^3}\right)\\
\frac{-d^7T}{5040dx^7} &= \pi^8\left(\frac{q^6}{p^8} + \frac{5q^4}{3p^6} + \frac{11q^2}{15p^4} + \frac{17}{315pp}\right)\\
\frac{d^8T}{40320dx^8} &= \pi^9\left(\frac{q^7}{p^9} + \frac{6q^5}{3p^7} + \frac{6q^3}{5p^5} + \frac{62q}{315p^3}\right)\\
&\text{etc.}
\end{aligned}$$

§40 Except these series we found several others in the *Introductio* from whom which in similar manner by differentiations others can be extracted.

For, we showed that it is

$$\frac{1}{2x} - \frac{\pi\sqrt{x}}{2x\tan\pi\sqrt{x}} = \frac{1}{1-x} + \frac{1}{4-x} + \frac{1}{9-x} + \frac{1}{16-x} + \frac{1}{25-x} + \text{etc.}$$

Let us put that the sum of this series is $= S$ that it is

$$S = \frac{1}{2x} - \frac{\pi}{2\sqrt{x}} \cdot \frac{\cos \pi\sqrt{x}}{\sin \pi\sqrt{x}};$$

it will be

$$\frac{dS}{dx} = -\frac{1}{2xx} + \frac{\pi}{4x\sqrt{x}} \cdot \frac{\cos \pi\sqrt{x}}{\sin \pi\sqrt{x}} + \frac{\pi\pi}{4x(\sin \pi\sqrt{x})^2},$$

which expression therefore yields the sum of this series

$$\frac{1}{(1-x)^2} + \frac{1}{(4-x)^2} + \frac{1}{(9-x)^2} + \frac{1}{(16-x)^2} + \frac{1}{(25-x)^2} + \text{etc.}$$

Further, we also showed that it is

$$\frac{\pi}{2\sqrt{x}} \cdot \frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1} - \frac{1}{2x} = \frac{1}{1+x} + \frac{1}{4+x} + \frac{1}{9+x} + \frac{1}{16+x} + \text{etc.}$$

If therefore this sum is put $= S$, it will be

$$\frac{-dS}{dx} = \frac{1}{(1+x)^2} + \frac{1}{(4+x)^2} + \frac{1}{(9+x)^2} + \frac{1}{(16+x)^2} + \text{etc.}$$

But it is

$$\frac{dS}{dx} = \frac{-\pi}{4x\sqrt{x}} \cdot \frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1} - \frac{\pi\pi}{x} \cdot \frac{e^{2\pi\sqrt{x}}}{(e^{2\pi\sqrt{x}}-1)^2} + \frac{1}{2xx}.$$

Therefore, the sum of this series will be

$$\frac{-dS}{dx} = \frac{\pi}{4x\sqrt{x}} \cdot \frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1} + \frac{\pi\pi}{x} \cdot \frac{e^{2\pi\sqrt{x}}}{(e^{2\pi\sqrt{x}}-1)^2} - \frac{1}{2xx}.$$

And in similar ways by means of further differentiations the sums of the following powers will be found.

§41 If the value of a certain product composed of factors involving the undetermined letter x was known, one will be able to find innumerable summable series from it by means of the same method. For, let the value of this product

$$(1+\alpha x)(1+\beta x)(1+\gamma x)(1+\delta x)(1+\varepsilon x)\text{etc.}$$

be $= S$, a function of x, of course; by taking logarithms it will be

$$\ln S = \ln(1+\alpha x) + \ln(1+\beta x) + \ln(1+\gamma x) + \ln(1+\delta x) + \text{etc.}$$

Now, take the differentials; after division by dx it will be

$$\frac{dS}{Sdx} = \frac{\alpha}{1+\alpha x} + \frac{\beta}{1+\beta x} + \frac{\gamma}{1+\gamma x} + \frac{\delta}{1+\delta x} + \text{etc.},$$

from the further differentiation of which the sums of any powers of these fractions will be found, precisely as we explained in more detail in the preceding examples.

§42 But, in the *Introductio* we exhibited several expressions of such a kind, to which we want to apply this method. If π is the arc of 180° of the circle whose radius is $= 1$, we showed that it is

$$\sin\frac{m\pi}{2n} = \frac{m\pi}{2n} \cdot \frac{4nn-mm}{4nn} \cdot \frac{16nn-mm}{16nn} \cdot \frac{26nn-mm}{36nn} \cdot \text{etc.}$$

$$\cos\frac{m\pi}{2n} = \frac{nn-mm}{nn} \cdot \frac{9nn-mm}{9nn} \cdot \frac{25nn-mm}{25nn} \cdot \frac{49nn-mm}{49nn} \cdot \text{etc.}$$

Let us put $n = 1$ and $m = 2x$ that it is

$$\sin \pi x = \pi x \cdot \frac{1-xx}{1} \cdot \frac{4-xx}{4} \cdot \frac{9-xx}{9} \cdot \frac{16-xx}{16} \cdot \text{etc.}$$

or

$$\sin \pi x = \pi x \cdot \frac{1-x}{1} \cdot \frac{1+x}{1} \cdot \frac{2-x}{2} \cdot \frac{2+x}{2} \cdot \frac{3-x}{3} \cdot \frac{3+x}{3} \cdot \frac{4-x}{4} \cdot \text{etc.}$$

and

$$\cos \pi x = \frac{1-4xx}{1} \cdot \frac{9-4xx}{9} \cdot \frac{25-4xx}{25} \cdot \frac{49-4xx}{49} \cdot \text{etc.}$$

or

$$\cos \pi x = \frac{1-2x}{1} \cdot \frac{1+2x}{1} \cdot \frac{3-2x}{3} \cdot \frac{3+2x}{3} \cdot \frac{5-2x}{5} \cdot \frac{5+2x}{5} \cdot \text{etc.}$$

Therefore, from these expressions, if one takes logarithms, it will be

$$\ln \sin \pi x = \ln \pi x + \ln \frac{1-x}{1} + \ln \frac{1+x}{1} + \ln \frac{2-x}{2} + \ln \frac{2+x}{2} + \ln \frac{3-x}{3} + \text{etc.}$$

$$\ln \cos \pi x = \ln \frac{1-2x}{1} + \ln \frac{1+2x}{1} + \ln \frac{3-2x}{3} + \ln \frac{3+2x}{3} + \ln \frac{5-2x}{5} + \text{etc.}$$

§43 Now, let us take the differentials of these series of logarithms and having divided by dx everywhere the first series will give

$$\frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

which is the series itself we treated in § 37. The other series on the other hand will give

$$\frac{-\pi \sin \pi x}{\cos \pi x} = -\frac{2}{1-2x} + \frac{2}{1+2x} - \frac{2}{3-2x} + \frac{2}{3+2x} - \frac{2}{5-2x} + \text{etc.}$$

Let us put $2x = z$ that it is $x = \frac{z}{2}$ and divide by -2; it will be

$$\frac{\pi \sin \frac{1}{2}\pi z}{2 \cos \frac{1}{2}\pi z} = \frac{1}{1-z} - \frac{1}{1+z} + \frac{1}{3-z} - \frac{1}{3+z} + \frac{1}{5-z} - \text{etc.}$$

But because it is

$$\sin \frac{1}{2}\pi z = \sqrt{\frac{1-\cos \pi z}{2}} \quad \text{and} \quad \cos \frac{1}{2}\pi z = \sqrt{\frac{1+\cos \pi z}{2}},$$

it will be

$$\frac{\pi\sqrt{1-\cos \pi z}}{\sqrt{1+\cos \pi z}} = \frac{2}{1-z} - \frac{2}{1+z} + \frac{2}{3-z} + \frac{2}{5-z} - \text{etc.}$$

or by writing x instead of z it will be

$$\frac{\pi\sqrt{1-\cos \pi x}}{\sqrt{1+\cos \pi x}} = \frac{2}{1-x} - \frac{2}{1+x} + \frac{2}{3-x} - \frac{2}{3+x} + \frac{2}{5-x} - \text{etc.}$$

Add this series to the one found first

$$\frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

and one will find the sum of this series

$$\frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \frac{1}{3+x} - \text{etc.}$$

to be $= \frac{\pi\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi}} + \frac{\pi\cos\pi x}{\sin\pi x}$. But this fraction $\frac{\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}}$, if the numerator and denominator are multiplied by $\sqrt{1-\cos\pi x}$, goes over into $\frac{1-\cos\pi x}{\sin\pi}$. Therefore, the sum of the series will be $= \frac{\pi}{\sin\pi x}$, which is the series itself we had in § 34; hence, we will not prosecute this any further.